

## The Adomian Decomposition Method For Boundary Eigenvalue Problems

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**Abstract:** The adomian decomposition method is a powerful method which consider the approximate solution of a nonlinear equation as an infinite series usually converges to the exact solution. In this paper, this method is proposed to solve some eigenvalue problems. three test problems are considered to illustrate the proposed decomposition method. It is shown that the series solutions converge to the exact solutions for each problem. Then we can obtain the eigenvalues of these problems.

**Key words:** Adomian decomposition method; eigenvalue problems.

### INTRODUCTION

The adomian decomposition method(ADM) firstly has been introduced by George Adomian in 1981 and developed in (Adomian, G., 1988). This method has been applied to solve differential and integral equations of linear and nonlinear problems in mathematics, physics, biology and chemistry and up to now a large number of research papers have been published to show the feasibility of the decomposition method(see for example(Wazwaz, A.M., 2000)).

The ADM decomposes a solution in to an infinite series which converges rapidly to the exact solution. The convergence of the ADM has been investigated by a number of authors (Abbaoui, K., *et al.*, 2001; Cherruault, Y., 1989; Cherruault, Y., G. Adomian, 1993; Mustafa Inc, 2005). This method can be applied directly for all types of differential and integral equations, linear or nonlinear, with constant or variable coefficients. The nonlinear problems are solved easily and elegantly without linearizing the problem by using ADM. The technique is capable of greatly reducing the size of computation work while still it provides an efficient numerical solution with high accuracy. It also avoids linearization, perturbation and discretization unlike other classical techniques.

#### **The Adomian Decomposition Method:**

Here the review of the standard adomian decomposition method is presented. We consider the differential equation

$$Ly + Ry + N y = g(x),$$

where N is a nonlinear operator, L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order less than L. Thus we get

$$Ly = g(x) - R(y) - N (y). \quad (2.1)$$

Operating with the operator  $L^{-1}$  on both sides of (2.1) we have

$$L^{-1} Ly = f(x) - L^{-1} R(y) - L^{-1} N (y), \quad (2.2)$$

where f is the function of  $L^{-1} g$  and constants of integration. For example if L is a second-order operator,  $L^{-1}$  is a twofold integration operator, then Eq.(2.1) for y yields,

$$y = a + bx + L^{-1} g - L^{-1} R(y) - L^{-1} N (y).$$

The adomian decomposition method introduces the solution  $y(x)$  by an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (2.3)$$

and the nonlinear operator  $N$  by an infinite series of polynomials

$$N(y) = \sum_{n=0}^{\infty} A_n, \quad (2.4)$$

where  $A_n$  are polynomials of  $y_0, \dots, y_n$  (Adomian, G., 1983; Adomian, G., 1986; Adomian, G., 1989) given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

The components  $y_n(x)$  of the solution  $y(x)$  will be determined recurrently and  $A_n$  are the adomian polynomials that can be constructed for various classes of nonlinearity according to specific algorithms.

Substituting (2.3) and (2.4) into (2.2) yields

Thus, we can identify

$$\sum_{n=0}^{\infty} y_n = f(x) - L^{-1} R \left( \sum_{n=0}^{\infty} y_n \right) - L^{-1} \left( \sum_{n=0}^{\infty} A_n \right).$$

$$y_0 = f(x),$$

$$y_1 = -L^{-1} R(y_0) - L^{-1} (A_0),$$

$\vdots$

$$y_{n+1} = -L^{-1} R(y_n) - L^{-1} (A_n).$$

We then define the  $n$ -term approximate to the solution  $y$  by with

$$\Phi_n[y] = \sum_{i=0}^n y_i,$$

$$\lim_{n \rightarrow \infty} \Phi_n[y] = y$$

### 3 Illustrative examples

#### Example 1:

Let us consider the differential equation

$$y'' + \lambda y = 0, \quad (3.5)$$

with the conditions

$$y(-a) = y(a), \quad y'(-a) = y'(a),$$

where  $a$  is a constant. Applying the Adomian decomposition method, Eq.(3.5) can be written as

$$Ly = -\lambda y, \quad (3.6)$$

where  $L = \frac{d^2}{dx^2}$  is the differential operator. Operating on both sides of Eq.(3.6) with the inverse operator of  $L$  (namely  $L^{-1}[\cdot] = \int_0^x \int_0^x [\cdot] dx dx$ ) yields  
 $y(x) = c_1 + c_2 x - L^{-1} \lambda y,$

$c_1, c_2$  are constants of integration. Using (2.3) it follows that

$$\sum_{n=0}^{\infty} y_n = c_1 + c_2 x - L^{-1} \lambda \sum_{n=0}^{\infty} y_n$$

Thus we can write

$$y_0 = c_1 + c_2 x,$$

$$y_{n+1} = -\lambda L^{-1} y_n, \quad n = 0, 1, 2, \dots,$$

from the condition  $y(-a) = y(a)$ , we have  $c_2 = 0$  and so  $y_0 = c_1$ . Then we have

$$y_1(x) = -\lambda L^{-1}(y_0) = -\lambda c_1 \frac{x^2}{2!},$$

$$y_2(x) = -\lambda L^{-1}(y_1) = \lambda^2 c_1 \frac{x^4}{4!},$$

$$y_3(x) = -\lambda L^{-1}(y_2) = -\lambda^3 c_1 \frac{x^6}{6!},$$

and so on ....

Considering these components, the solution can be approximated as

$$y(x) = \Phi_n[y] = \sum_{i=0}^n y_i(x),$$

$$\Phi_1 = c_1 - \lambda c_1 \frac{x^2}{2!},$$

$\Phi_n$  contains the exact power series expansion of the closed form solution

$$\Phi_2 = c_1 - \lambda c_1 \frac{x^2}{2!} + \lambda^2 c_1 \frac{x^4}{4!},$$

$$\Phi_3 = c_1 - \lambda c_1 \frac{x^2}{2!} + \lambda^2 c_1 \frac{x^4}{4!} - \lambda^3 c_1 \frac{x^6}{6!},$$

$\vdots$

$$y(x) = c_1 \cos(\sqrt{\lambda} x), \quad c_1 \neq 0.$$

Using the other condition  $y'(-a) = y'(a)$ , the eigenvalues are computed exactly

$$\lambda_n = \left( \frac{n\pi}{a} \right)^2, \quad n = 0, 1, 2, \dots$$

### Example 2:

We consider the BVP

$$y^{(4)} + \lambda y'' = 0, \tag{3.7}$$

$$\begin{aligned} y(0) &= 0, & y''(0) &= 0, \\ y(1) &= 0, & y'(1) &= 0. \end{aligned}$$

Applying the Adomian decomposition method, Eq.(3.7) can be written as

$$Ly = -\lambda y'' \quad (3.8)$$

where  $L = \frac{d^4}{dx^4}$  is the differential operator. Operating  $L^{-1} (L^{-1}[\cdot] = \int_0^x \int_0^x \int_0^x \int_0^x [\cdot] dx dx dx dx)$  on both sides of Eq.(3.8) yields

$$y(x) = a \frac{x^3}{3!} + b \frac{x^2}{2!} + cx + d - L^{-1}\lambda y''$$

where a, b, c, d are constants of integration. Using (2.3) it follows that

$$\sum_{n=0}^{\infty} y_n = a \frac{x^3}{3!} + b \frac{x^2}{2!} + cx + d - L^{-1}\lambda \left( \sum_{n=0}^{\infty} y_n \right)''$$

thus we can write

$$y_0 = a \frac{x^3}{3!} + b \frac{x^2}{2!} + cx + d$$

$$y_{n+1} = -\lambda L^{-1} y_n'', \quad n = 0, 1, 2, \dots$$

from the conditions  $y(0) = 0, y''(0) = 0$ , we find that  $b = 0, d = 0$  and so  $y_0 = a \frac{x^3}{3!} + cx$ .

Then we have

and so on ....

$$\begin{aligned} y_1(x) &= -\lambda L^{-1}(y_0'') \\ &= -\lambda a \frac{x^5}{5!}, \\ y_2(x) &= -\lambda L^{-1}(y_1'') \\ &= \lambda^2 a \frac{x^7}{7!}, \\ y_3(x) &= -\lambda L^{-1}(y_2'') \\ &= -\lambda^3 a \frac{x^9}{9!}, \end{aligned}$$

Considering these components, the solution can be approximated as

$$y(x) = \Phi_n[y] = \sum_{i=0}^n y_i(x),$$

$$\Phi_1 = cx + a \left( \frac{x^3}{3!} - \lambda \frac{x^5}{5!} \right),$$

$$\Phi_2 = cx + a\left(\frac{x^3}{3!} - \lambda \frac{x^5}{5!} + \lambda^2 \frac{x^7}{7!}\right),$$

$$\Phi_3 = cx + a\left(\frac{x^3}{3!} - \lambda \frac{x^5}{5!} + \lambda^2 \frac{x^7}{7!} - \lambda^3 \frac{x^9}{9!}\right),$$

⋮

$\Phi_n$  contains the exact power series expansion of the closed form solution

$$y(x) = cx - \frac{a}{\lambda\sqrt{\lambda}}(\sin \sqrt{\lambda}x),$$

Using the conditions  $y(1) = 0$ ,  $y'(1) = 0$ , the eigenvalues are computed with the equation

$$\tan(\sqrt{\lambda}) = \sqrt{\lambda}$$

Using iteration methods the zeros of this equation in interval (0,20) are

$$0 \quad 4.4934 \quad 7.7253 \quad 10.9041 \quad 14.0662 \quad 17.2208$$

### Example 3:

In this example we consider the differential equation

$$y'' + \lambda y = 0, \quad 0 < x < \infty, \quad (3.9)$$

with the conditions

Similarly to Example 1 we have

$$y(0) = 0,$$

$y$  and  $y'$  are finite as  $x \rightarrow \infty$ .

$$y(x) = c_1 + c_2 x - L^{-1} \lambda y,$$

$c_1, c_2$  are constants of integration. Applying ADM technique yields

$$\sum_{n=0}^{\infty} y_n = c_1 + c_2 x - L^{-1} \lambda \sum_{n=0}^{\infty} y_n$$

Thus we can obtain

$$y_0 = c_1 + c_2 x,$$

$$y_{n+1} = -\lambda L^{-1} y_n, \quad n = 0, 1, 2, \dots,$$

from the condition  $y(0) = 0$ , we have  $c_1 = 0$ , then  $y_0 = c_2 x$ . Therefore we have

and so on ....

$$\begin{aligned} y_1(x) &= -\lambda L^{-1}(y_0) \\ &= -\lambda c_2 \frac{x^3}{3!}, \end{aligned}$$

$$\begin{aligned}
 y_2(x) &= -\lambda L^{-1}(y_1) \\
 &= \lambda^2 c_2 \frac{x^5}{5!}, \\
 y_3(x) &= -\lambda L^{-1}(y_2) \\
 &= -\lambda^3 c_2 \frac{x^7}{7!},
 \end{aligned}$$

Considering these components, the solution can be approximated as

$$y(x) = \Phi_n[y] = \sum_{i=0}^n y_i(x),$$

$$\Phi_1 = c_2 x - \lambda c_2 \frac{x^3}{3!},$$

$$\Phi_2 = c_2 x - \lambda c_2 \frac{x^3}{3!} + \lambda^2 c_2 \frac{x^5}{5!},$$

$$\Phi_3 = c_2 x - \lambda c_2 \frac{x^3}{3!} + \lambda^2 c_2 \frac{x^5}{5!} - \lambda^3 c_2 \frac{x^7}{7!},$$

⋮

contains the exact power series expansion of the closed form solution

$$y(x) = \frac{c_2}{\sqrt{\lambda}} \sin(\sqrt{\lambda} x), \quad c_1 \neq 0.$$

Since  $y, y'$  when  $x \rightarrow \infty$  are finite, so  $\lambda$  can be each positive real value.

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